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Logarithmic matrix norms in motion stability problems☆

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ABSTRACT

The problem of the stability of the motions of mechanical systems, described by non-linear non-autonomous systems of ordinary differential equations, is considered. Using the logarithmic matrix norm method, and constructing a reference system, the sufficient conditions for the asymptotic and exponential stability of unperturbed motion and for the stabilization of progammed motions of such systems are obtained. The problem of the asymptotic stability of a non-conservative system with two degrees of freedom is solved, taking for parametric disturbances into account. Examples of the solution of the problem of stabilizing programmed motions – for an inverted double pendulum and for a two-link manipulator on a stationary base – are considered.

1. Problem formulation

Suppose R^n is an n-dimensional real linear vector space with norm $|\cdot|$, $R^{n\times n}$ is a linear, active, square-matrix space and $\mathbf{I}\in R^{n\times n}$ is the identity matrix.

Definition 1 *([1])*. The operator norm $\|\mathbf{A}\|$ of matrix $\mathbf{A}\in R^{n\times n}$, subject to the vector norm $|\cdot|$, is

$$\|\mathbf{A}\| = \max_{\mathbf{x}\in R^n,\,|\mathbf{x}|=1} |\mathbf{A}\mathbf{x}| = \max_{\mathbf{x}\neq 0}\frac{|\mathbf{A}\mathbf{x}|}{|\mathbf{x}|}$$

Definition 2 *([2])*. The logarithmic norm $\gamma(\mathbf{A})$ of the matrix $\mathbf{A}\in R^{n\times n}$ is

$$\gamma(\mathbf{A}) = \lim_{h\to+0}\frac{1}{h}[\|\mathbf{I}+h\mathbf{A}\|-1]$$

Remark. The logarithmic norm of the matrix can take negative values and is not a matrix norm in the classic sense.

The operator norm of the fundamental matrix $\mathbf{\Phi}(\tau, t)$ of solutions of the linear system

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$$

satisfies the inequality[2]

$$\|\mathbf{\Phi}(\tau, t)\| \le \exp\left(\int_\tau^t \gamma(\mathbf{A}(s))ds\right)$$

which enables us, using vector-norm-type Lyapunov functions, to construct the reference equations

$$\dot{u} = \gamma(\mathbf{A}(t))u$$

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An analogous reference equation can be constructed by considering the problem of the stability of the zero solution for a non-linear system:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{f}(t, \mathbf{0}) = \mathbf{0}; \quad (t, \mathbf{x}) \in S \subset R^{+} \times R^{n}$$

where $\mathbf{f}(t, \mathbf{x})$ is a function that is continuous with respect to the set of arguments and continuously differentiable with respect to $\mathbf{x}$. Denoting by $\mathbf{F}(t, \mathbf{x})$ the Jacobi matrix of the function $\mathbf{f}(t, \mathbf{x})$, $\mathbf{F}(t, \mathbf{x})=\partial\mathbf{f}(t, \mathbf{x})/\partial\mathbf{x}$, and choosing one of the vector norms as the Lyapunov function V, $V=|\mathbf{x}|$, we obtain a reference equation of the form

$$\dot{u} = \left(\int_0^1 \gamma(\mathbf{F}(t, s\mathbf{x}(t)))ds\right)u$$

However, if the system has the form

$$\dot{\mathbf{x}}(t) = \mathbf{F}_1(t, \mathbf{x})\mathbf{x}$$

then it is possible to construct a reference equation of the form

$$\dot{u} = \gamma(\mathbf{F}_1(t, \mathbf{x}(t)))u$$

Consider the problem of the stability of the zero solution of the system

$$\ddot{\mathbf{x}} + \mathbf{A}(t, \mathbf{x}, \dot{\mathbf{x}})\dot{\mathbf{x}} + \mathbf{B}(t, \mathbf{x})\mathbf{x} = \mathbf{0}, \quad \mathbf{x} \in R^{n}, \quad t \in R \tag{1.1}$$

where $\mathbf{A}(t, \mathbf{x}, \mathbf{y})$ and $\mathbf{B}(t, \mathbf{x})$ are $n \times n$ matrices with uniformly continuous constrained elements, where here the matrix $\mathbf{B}(t, \mathbf{x})$ is non-degenerate, $\det\mathbf{B}(t, \mathbf{x}) \geq d_0 = \text{const} > 0$.

By replacing the variables

$$\mathbf{x}_1 = \mathbf{x}, \quad \mathbf{x}_2 = \mathbf{x} + c^{-1}\dot{\mathbf{x}}; \quad c = \text{const} > 0$$

we transform system (1.1) to the form

$$\begin{aligned}
\dot{\mathbf{x}}_1 &= -c\mathbf{x}_1 + c\mathbf{x}_2 \\
\dot{\mathbf{x}}_2 &= (-c\mathbf{I} + \mathbf{A}(t, \mathbf{x}_1, -c\mathbf{x}_1 + c\mathbf{x}_2) - c^{-1}\mathbf{B}(t, \mathbf{x}_1))\mathbf{x}_1 + \\
&+ (c\mathbf{I} - \mathbf{A}(t, \mathbf{x}_1, -c\mathbf{x}_1 + c\mathbf{x}_2))\mathbf{x}_2
\end{aligned} \tag{1.2}$$

If, for system (1.2), we adopt a Lyapunov vector function of the form

$$\mathbf{V} = (V_1, V_2)', \quad V_1 = |\mathbf{x}_1|, \quad V_2 = |\mathbf{x}_2|$$

where $|\cdot|$ is a certain vector norm, we can construct a generalized reference system consisting of Eq. (1.2) and the equations

$$\begin{aligned}
\dot{u}_1 &= -cu_1 + cu_2 \\
\dot{u}_2 &= \|-c\mathbf{I} + \mathbf{A}(t, \mathbf{x}_1, -c\mathbf{x}_1 + c\mathbf{x}_2) - c^{-1}\mathbf{B}(t, \mathbf{x}_1)\|u_1 + \\
&+ (c + \gamma(-\mathbf{A}(t, \mathbf{x}_1, -c\mathbf{x}_1 + c\mathbf{x}_2)))u_2
\end{aligned} \tag{1.3}$$

where γ is the logarithmic norm and $\|\cdot\|$ is the operator norm of the corresponding matrix.

Using the classic comparison method approach,[3,4] from the properties of stability of the zero solution $\mathbf{x}_1 = \mathbf{x}_2 = 0$, $u_1 = u_2 = 0$ of reference system (1.3) with respect to the variables u_1 and u_2 we can conclude that there is an analogous property of stability of the zero solution

$$\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{0} \tag{1.4}$$

of system (1.2). The limit equation method enables us, in a number of cases to reduce the requirements concerning the reference system considerably: for example, for asymptotic stability of the zero solution of the system investigated it is not essential for the reference system to have an asymptotically stable zero solution.[5] On the basis of the principle of quasi-invariance of a positive limit set of perturbed motion, the sufficient conditions for the asymptotic stability of unperturbed motion were obtained[5] using of Lyapunov vector functions.

We will assume that, in relation to t, the right-hand side of system (1.2) uniformly satisfies the Lipschitz function with respect to ($\mathbf{x}_1$, $\mathbf{x}_2$). Then, for system (1.2) we can construct the family of limit systems[5]

$$\begin{aligned}
\dot{\mathbf{x}}_1 &= -c\mathbf{x}_1 + c\mathbf{x}_2 \\
\dot{\mathbf{x}}_2 &= (-c\mathbf{I} + \mathbf{A}^*(t, \mathbf{x}_1, -c\mathbf{x}_2 + c\mathbf{x}_2) - c^{-1}\mathbf{B}^*(t, \mathbf{x}_1))\mathbf{x}_1 + \\
&+ (c\mathbf{I} - \mathbf{A}^*(t, \mathbf{x}_1, -c\mathbf{x}_1 + c\mathbf{x}_2))\mathbf{x}_2
\end{aligned} \tag{1.5}$$

The asterisk denotes the limit function to the initial function.

We will consider the following problem: using the method of logarithmic matrix norms and results obtained earlier,[5] it is required to find the sufficient conditions for exponential and asymptotic stability of the zero solution of system (1.1).

2. Principal theorems of asymptotic and exponential stability

For $r = \text{const} > 0$ or $r = +\infty$, we will introduce the notation

$$D_r = \{(\mathbf{x}, \mathbf{y}) \in R^n \times R^n: |\mathbf{x}| < r, |\mathbf{y}| < r\}$$

Theorem 1. *For exponential stability of the zero solution*

$$\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0} \tag{2.1}$$

of system (1.1), it is sufficient to have the constants $k > 1$, $r > 0$, $c > 0$, and $L > 0$ and the continuous function ε: $R^+ \to R$, such that, for any $t_0 \geq 0$ and for all $t \geq t_0$, the following inequalities are satisfied:

$$\sup_{(\mathbf{x}, \mathbf{y}) \in D_r} \{c + k\|-c\mathbf{I} + \mathbf{A}(t, \mathbf{x}, \mathbf{y}) - c^{-1}\mathbf{B}(t, \mathbf{x})\| + \gamma(-\mathbf{A}(t, \mathbf{x}, \mathbf{y}))\} \leq \varepsilon(t)$$
$$\int_{t_0}^{t} M(\varepsilon(s))ds \leq L; \quad M(\varepsilon(s)) = \max\{\varepsilon(s), -c + c/k\} \tag{2.2}$$

Proof. According to the second condition of system (2.2), the zero solution (1.4) of system (1.2) is uniformly stable. In fact, we will adopt the Lyapunov function $V = \max\{|\mathbf{x}_1|, k|\mathbf{x}_2|\}$. Then, calculating its derivative by virtue of system (1.2), we obtain the estimate

$$\dot{V} \leq M(\varepsilon(t))V$$

From this, for all $t \geq t_0$, $t_0 \geq 0$ we obtain the following inequality

$$\max\{|\mathbf{x}_1(t)|, k|\mathbf{x}_2(t)|\} \leq \max\{|\mathbf{x}_1(t_0)|, k|\mathbf{x}_2(t_0)|\}\exp\left(\int_{t_0}^{t} M(\varepsilon(s))ds\right)$$

The reference equation $\dot{u} = M(\varepsilon(t))u$ will be uniformly stable according to the second condition of system (2.2), which means that the zero solution (1.4) of system (1.2) will be uniformly stable.

We will prove that solution (1.4) of system (1.2) is exponentially stable.

Using the theorem of localization of a positive limit set,[5] for solutions of the limit system (1.5) for all $t \geq 0$ we will have[5]

$$\max\{|\mathbf{x}_1^*(t)|, k|\mathbf{x}_2^*(t)|\} \leq \max\{|\mathbf{x}_1^*(0)|, k|\mathbf{x}_2^*(0)|\}\exp\left(\int_{0}^{t} M(\varepsilon^*(s))ds\right)$$

Analysing this equation, we establish that there is an instant of time t_1, $t_1 \geq 0$, such that

$$M(\varepsilon(t)) \leq -m = \text{const} < 0, \quad \forall t \geq t_1 + t_0 \tag{2.3}$$

In fact, for solutions of limit system (1.5) we will have

$$|\mathbf{x}_1^*(t)| \geq k|\mathbf{x}_2^*(t)|, \quad \forall t \geq 0 \tag{2.4}$$

since, in the opposite case, we will find that there is an instant $t^* \geq 0$ such that

$$M(\varepsilon^*(t^*))k|\mathbf{x}_2^*(t^*)| < \varepsilon^*(t^*)k|\mathbf{x}_2^*(t^*)|$$

and we arrive at the contradiction: $M(\varepsilon^*(t^*)) < \varepsilon^*(t^*)$.

From inequality (2.4) it follows that

$$M(\varepsilon^*(t)) \leq -\varepsilon_0 = \text{const} < 0, \quad \forall t \geq 0$$

and hence inequality (2.3) holds.

From inequality (2.3), for solutions of system (1.2) for all $t \geq t_0$ we will have

$$\max\{|\mathbf{x}_1(t)|, k|\mathbf{x}_2(t)|\} \leq C\max\{|\mathbf{x}_1(t_0)|, k|\mathbf{x}_2(t_0)|\}\exp(-m(t - t_0))$$

where

$$C = \exp(lt_1), \quad l = \sup_{t \geq 0} M(\varepsilon(t))$$

Thus, solution (1.4) of system (1.2) is exponentially stable. Hence, the zero solution (2.1) of system (1.1) is exponentially stable.

In the next theorem, uniform asymptotic stability of the zero solution of the initial system is achieved when the reference system is stable (non-asymptotically).

Theorem 2. *Let* $|\cdot|$ *be a spherical norm and*

$$c + \sup_{(\mathbf{x},\mathbf{y}) \in D_r} \gamma(-\mathbf{A}(t,\mathbf{x},\mathbf{y})) \le -\delta = \text{const} < 0 \tag{2.5}$$

Then, for uniform asymptotic stability of the zero solution (2.1) of system (1.1), it is sufficient for the conditions of the previous theorem to be satisfied when $k = 1$.

Proof. From the conditions of the theorem it follows that the zero solution (1.4) of system (1.2) is uniformly stable. Hence, the zero solution (2.1) of system (1.1) is uniformly stable.

We will show that solution (2.1) of system (1.1) is uniformly attractive.

Taking inequality (2.5) into account, we will establish that, to solve the limit system (1.5) for all $t \ge 0$, the inequality $|\mathbf{x}_1^*(t)| = |\mathbf{x}_2^*(t)| = m$. And, because $|\cdot|$ is a spherical vector norm, we can see from the latter equation that for all $t \ge 0$ the following relation holds

$$\left|\dot{\mathbf{x}}_1^*(t)\right|^2 = -cd\left|\mathbf{x}_1^*(t)\right|^2/dt = 0$$

Thus, for solutions of limit system (1.5), the equation $\dot{\mathbf{x}}_1^*(t) = \mathbf{0}$ is satisfied for all $t \ge 0$. Also, as the matrix $\boldsymbol{B}^*(t, \boldsymbol{x}_1)$ is non-singular, the latter equation is only possible if $\dot{\mathbf{x}}_1^*(t) \equiv \mathbf{0}$ for all $t \ge 0$, i.e. the zero solution (2.1) of system (1.1) is uniformly attractive.

Theorem 3. *For exponential stability of the zero solution (2.1) of system (1.1) it is sufficient for the following conditions to be satisfied*

$$\exists k > 1: \sup_{(\mathbf{x},\mathbf{y}) \in D_r} \{c + k\|-c\mathbf{I} + \mathbf{A}(t,\mathbf{x},\mathbf{y}) - c^{-1}\mathbf{B}(t,\mathbf{x})\| + \gamma(-\mathbf{A}(t,\mathbf{x},\mathbf{y}))\} \le \varepsilon(t)$$

$$\overline{\lim_{T \to \infty}} \frac{1}{T} \int_{t_0}^{t_0+T} M(\varepsilon(t))dt \le -m = \text{const} < 0$$

Proof. We will choose a Lyapunov function in the form of the rectangular vector norm $V = \{|\boldsymbol{x}_1|; k|\boldsymbol{x}_2|\}$. We will then have $\dot{V} \le M(\varepsilon(t))V$. From the condition of the theorem it follows that there is an instant of time $t_1 > t_0$ such that

$$M(\varepsilon(t)) \le -\delta = \text{const} < 0, \quad \forall t \ge t_1$$

From this we obtain the statement of the theorem.

Theorem 4. *For exponential stability of the zero solution (2.1) of system (1.1) it is sufficient for the following conditions to be satisfied*

$$\sup_{(\mathbf{x},\mathbf{y}) \in D_r} \max\{2c + \gamma(-\mathbf{A}(t,\mathbf{x},\mathbf{y})); -c + \|-c\mathbf{I} + \mathbf{A}(t,\mathbf{x},\mathbf{y}) - c^{-1}\mathbf{B}(t,\mathbf{x})\|\} \le \varepsilon(t)$$

$$\overline{\lim_{T \to \infty}} \frac{1}{T} \int_{t_0}^{t_0+T} \varepsilon(t)dt \le -m = \text{const} < 0$$

The proof is similar to the proof of Theorem 3 if the Lyapunov function is adopted in the form of the octahedral vector norm $V = |\mathbf{x}_1| + |\mathbf{x}_2|$. Theorems 3 and 4 were proved using well-known results.[8]

3. The stability of a non-conservative system with two degrees of freedom taking parametric perturbations into account

Let us consider the problem of the asymptotic stability of the zero solution of the system[6]

$$A_1\ddot{x} + b_1\dot{x} + H\dot{y} + c_1x + py = X, \quad A_2\ddot{y} + b_2\dot{y} - H\dot{x} + c_2y - px = Y \tag{3.1}$$

where $A_1 > 0$ and $A_2 > 0$ are generalized coefficients of inertia, $-b_1\dot{x}$ and $-b_2\dot{y}$ are dissipative forces, $-H\dot{y}$ and $H\dot{x}$ are gyroscopic forces, H is a parameter, $-c_1x$ and $-c_2y$ are potential forces, $-py$ and px are forces of radial correction and X and Y are terms containing $x, y, \dot{x}, \dot{y}$ to a power higher than 1.

We will assume that system (3.1) is subject to parametric perturbations, on account of which the proportionality factors of the dissipative and potential forces change with time. We will also assume that $b_i = b_i(t, x, y, \dot{x}, \dot{y})$ and $c_i = c_i(t, x, y, \dot{x}, \dot{y})$ are continuous functions satisfying the conditions

$$0 \le c_i^{\min} \le c_i \le c_i^{\max} = \text{const} > 0, \quad 0 < b_i^{\min} \le b_i \le b_i^{\max}$$

In system (3.1), we will replace the variables

$$\tilde{x} = \sqrt{A_1}x, \quad \tilde{y} = \sqrt{A_2}y$$

and introduce the notation

$$\tilde{b}_i = \frac{b_i}{A_i}, \quad \tilde{c}_i = \frac{c_i}{A_i}; \quad i = 1, 2$$

$$\tilde{H} = \frac{H}{\sqrt{A_1A_2}}, \quad \tilde{p} = \frac{p}{\sqrt{A_1A_2}}, \quad \tilde{X} = \frac{X}{\sqrt{A_1A_2}}, \quad \tilde{Y} = \frac{Y}{\sqrt{A_1A_2}}$$

Then, in the new variables, system (3.1) will take the form

$$\ddot{\tilde{x}} + \tilde{b}_1\dot{\tilde{x}} + \tilde{H}\dot{\tilde{y}} + \tilde{c}_1\tilde{x} + \tilde{p}\tilde{y} = \tilde{X}, \quad \ddot{\tilde{y}} + \tilde{b}_2\dot{\tilde{y}} - \tilde{H}\dot{\tilde{x}} + \tilde{c}_2\tilde{y} - \tilde{p}\tilde{x} = \tilde{Y} \tag{3.2}$$

We will define the matrices

$$\mathbf{A} = \left\| \begin{matrix} \tilde{b}_1 & \tilde{H} \\ -\tilde{H} & \tilde{b}_2 \end{matrix} \right\|, \quad \mathbf{B} = \left\| \begin{matrix} \tilde{c}_1 & \tilde{p} \\ -\tilde{p} & \tilde{c}_2 \end{matrix} \right\|$$

and adopt the number $c = p/H$. Calculating the corresponding norms, according to Theorem 2 we will write the conditions of uniform asymptotic stability of the zero solution $x = y = \dot{x} = \dot{y} = 0$ of system (3.1) in the form

$$b_i^{\min} \geq \frac{A_i p}{H} + \frac{Hc_i^{\max}}{2p}, \quad i = 1, 2, \quad \frac{b_1^{\min}}{A_1} + \frac{b_2^{\min}}{A_2} \geq \frac{2p}{H} + \frac{H}{p}\max\left\{\frac{c_1^{\max}}{A_1}, \frac{c_2^{\max}}{A_2}\right\}$$

$$\frac{b_j^{\max}}{A_j} - \frac{b_k^{\min}}{A_k} \leq \frac{Hc_j^{\min}}{pA_j}, \quad j, k = 1, 2, \quad j \neq k \tag{3.3}$$

In the case where

$$A_1 = A_2 = A, \quad b_1^{\min} = b_2^{\min} = b_{\min}, \quad b_1^{\max} = b_2^{\max} = b_{\max}$$

$$c_1^{\min} = c_2^{\min} = c_{\min}, \quad c_1^{\max} = c_2^{\max} = c_{\max}$$

conditions (3.3) acquire the more compact form

$$b_{\min} \geq \frac{Ap}{H} + \frac{c_{\max}H}{2p}, \quad b_{\max} - b_{\min} \leq \frac{c_{\min}H}{p}$$

Example 1. The Problem of the stable functioning of a vertical gyroscope with radial correction. The equations of motion of the axis of a vertical gyroscope have the form[6]

$$A\ddot{\alpha} + b\dot{\alpha} - H\dot{\beta} - k\beta = X_1, \quad A\ddot{\beta} + b\dot{\beta} + H\dot{\alpha} + k\alpha = X_2 \tag{3.4}$$

where A is the equatorial moment of inertia of the gyroscope, b is the coefficient of the resistance forces, H is the angular momentum, k is the slope of the characteristic curve of the torque sensors, and X_1 and X_2 are non-linear functions of α, β, $\dot{\alpha}$ and $\dot{\beta}$.

We will assume that $b = b(t, \alpha, \beta, \dot{\alpha}, \dot{\beta})$ is a certain constrained, uniformly continuous function, where

$$0 < b_0 \leq b(t, \alpha, \beta, \dot{\alpha}, \dot{\beta}) \leq b_1 = \text{const} > 0$$

Then, conditions (3.3) of uniform asymptotic stability of the zero solution $\alpha = \beta = \dot{\alpha} = \dot{\beta} = 0$ reduce to the inequality

$$b_0 > Ak/H$$

which, if b = constant > 0, is identical with the Routh–Hurwitz condition.

We will now consider the problem of the stability of the zero solution of the system

$$\ddot{x} + b\dot{x} + cx + py = X, \quad \ddot{y} + b\dot{y} + cy - px = Y \tag{3.5}$$

The problem of stabilizing the transverse motion of a rotor rotating in an aerodynamic medium by fitting an annular damper[7], reduces to this system.

In system (3.5) we will take into account the action of parametric disturbances and assume that the coefficients b, c, and p are functions of t, x, y, $\dot{x}$, $\dot{y}$ and vary with time within certain finite limits:

$$0 < b_{\min} \leq b \leq b_{\max}, \quad 0 < p_{\min} \leq p \leq p_{\max}, \quad 0 < c_{\min} \leq c \leq c_{\max}$$

Then, applying Theorem 2, we obtain the following condition of uniform asymptotic stability of the zero solution $x = y = \dot{x} = \dot{y} = 0$ of system (3.5)

$$b_{\min}^2 \geq 2(c_i + c_i^{-1} p_{\max}^2), \quad i = \max, \min$$

4. Stabilization of the programmed motion of mechanical systems

Let us consider the problem of stabilizing the programmed motion $\mathbf{q} = \mathbf{q}_0(t)$ of a mechanical system with n degrees of freedom

$$\mathbf{H}(t, \mathbf{q})\ddot{\mathbf{q}} + \mathbf{F}(t, \mathbf{q}, \dot{\mathbf{q}}) = \mathbf{u} \tag{4.1}$$

where $\mathbf{q} \in R^n$ is the vector of generalized coordinates, $\dot{\boldsymbol{q}}$ is the vector of generalized velocities, $\mathbf{H}(t, \mathbf{q})$ is an $n \times n$ matrix, $\boldsymbol{F}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is an $n \times 1$ vector with constrained uniformly continuous elements, $\boldsymbol{u} = \tilde{\boldsymbol{u}} + \boldsymbol{u}_0(t)$ is the vector of control actions, $\tilde{\boldsymbol{u}}$ is the additional control action and $\mathbf{u}_0(t)$ is the open-loop control.

We will use $\mathbf{x} = \mathbf{q} - \mathbf{q}_0(t)$ to denote the deviation of the true motion from the programmed motion and write the linearized equations in deviations

$$\ddot{\mathbf{x}} + \mathbf{A}(t)\dot{\mathbf{x}} + \mathbf{B}(t)\mathbf{x} = \mathbf{C}(t)\tilde{\mathbf{u}} \tag{4.2}$$

where

$$\mathbf{A}(t) = \|a_{ij}(t)\|, \quad a_{ij}(t) = \frac{\partial L_i}{\partial \dot{q}_j}(t, \mathbf{q}_0(t), \dot{\mathbf{q}}_0(t))$$

$$\mathbf{B}(t) = \|b_{ij}(t)\|, \quad b_{ij}(t) = \frac{\partial L_i}{\partial q_j}(t, \mathbf{q}_0(t), \dot{\mathbf{q}}_0(t)); \quad i, j = 1, \ldots, n$$

$$\mathbf{C}(t) = \mathbf{H}^{-1}(t, \mathbf{q}_0(t))$$

while the vector with components L_i is defined by the formula

$$\mathbf{L}(t, \mathbf{q}, \dot{\mathbf{q}}) = \mathbf{H}^{-1}(t, \mathbf{q})\mathbf{F}(t, \mathbf{q}, \dot{\mathbf{q}}) - \mathbf{H}^{-1}(t, \mathbf{q})\mathbf{u}_0(t)$$

We will assume that only the coordinates of the object are available for measurement, while the vector of the additional control action is defined as a linear function of the coordinates:

$$\tilde{\mathbf{u}} = \mathbf{K}\mathbf{x}$$

where $\mathbf{K}$ is a certain constant matrix.

For the continuous function a: $R^+ \to R$ we will introduce the notation

$$[a(t)]_+ = \max\{a(t), 0\}, \quad \forall t \geq 0$$

The following theorems concerning the stabilization of programmed motion will then hold.

Theorem 5. *Suppose be constants $k > 1$ and $d = \text{const} > 0$ exist, such that*

$$\int_0^\infty [d + k\|-d\mathbf{I} + \mathbf{A}(s) - d^{-1}\mathbf{B}(s) + d^{-1}\mathbf{C}(s)\mathbf{K}\| + \gamma(-\mathbf{A}(s))]_+ ds < +\infty$$

Then the control

$$\mathbf{u} = \tilde{\mathbf{u}}_0 + \mathbf{u}_0(t), \quad \tilde{\mathbf{u}} = \mathbf{K}(\mathbf{q} - \mathbf{q}_0(t)) \tag{4.3}$$

will stabilize the programmed motion $\mathbf{q}_0(t)$ of system (4.1) and the zero position of equilibrium $\boldsymbol{x} = \dot{\boldsymbol{x}} = \mathbf{0}$ of system (4.2) will be exponentially stable.

Theorem 6. *Let $|\cdot|$ be a spherical vector norm and*

$$d + \gamma(-\mathbf{A}(t)) \leq -\delta = \text{const} < 0$$

and let the conditions of the previous theorem also be satisfied for $k = 1$.

Then, the control (4.3) will stabilize the programmed motion $\mathbf{q}_0(t)$ of system (4.1) and the zero position of equilibrium $\boldsymbol{x} = \dot{\boldsymbol{x}} = \mathbf{0}$ of system (4.2) will be uniformly asymptotically stable.

Theorems 5 and 6 supplement well-known results[9,10] on the stabilization of programmed motions of mechanical systems.

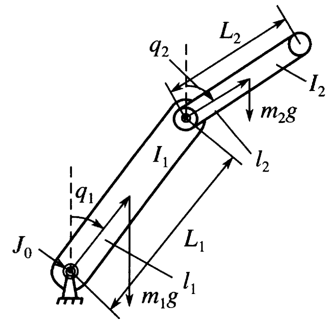

Fig. 1.

Example 2. The problem of stabilizing the programmed motion of an inverted double pendulum. The controlled motion of an inverted double pendulum (Fig. 1) is described by the equation[11]

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{P}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{W}(\mathbf{q})\mathbf{q} = \mathbf{u} \tag{4.4}$$

where [11]

$$\mathbf{H}(\mathbf{q}) = \begin{Vmatrix} J_0 + I_1 + m_1 l_1^2 + m_2 L_1^2 & m_2 L_1 l_2 \cos(q_1 - q_2) \\ m_2 L_1 l_2 \cos(q_1 - q_2) & m_2 l_2^2 + I_2 \end{Vmatrix}$$

$$\mathbf{P}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{Vmatrix} k & \dot{q}_2 m_2 L_1 l_2 \sin(q_1 - q_2) \\ -\dot{q}_1 m_2 L_1 l_2 \sin(q_1 - q_2) & k \end{Vmatrix}$$

$$\mathbf{W}(\mathbf{q}) = \begin{Vmatrix} -g(m_1 l_1 + m_2 L_1) q_1^{-1} \sin q_1 & 0 \\ 0 & -g m_2 l_2 q_2^{-1} \sin q_2 \end{Vmatrix}$$

$\mathbf{q} = (q_1, q_2)'$ is the vector of the angles between the vertical axis and the corresponding link of the pendulum, $\mathbf{u} = (u_1, u_2)'$ is the vector of control moments, a moment u_1 being applied to the lower link and a moment u_2 to the upper link, L_1 is the length of the first link, m_i is the mass of a pendulum link, I_i is the moment of inertia of a link, l_i is the distance from the centre of gravity of a link to the point of support, here $i = 1$ corresponds to the lower link and $i = 2$ to the upper link, J_0 is the moment of inertia of the drive shaft, g is the acceleration due to gravity and k is the coefficient of the moments of the resistance forces, linear with respect to the generalized forces arising during the motion of the links.

Let the programmed trajectory have the form

$$\mathbf{q}_0(t) = \begin{Vmatrix} \sin(2.8t) - 0.5 \\ \cos(3.9t) - 0.5 \end{Vmatrix}$$

and let the mechanical parameters of the pendulum be as follows:

$$m_1 = 0.145 \text{ Kg}, \quad m_2 = 0.07 \text{ Kg}, \quad L_1 = 0.3452 \text{ m}$$
$$l_1 = 0.1527 \text{ m}, \quad l_2 = 0.1609 \text{ m}, \quad I_1 = 0.0692 \text{ Kg m}^2, \quad I_2 = 0.0414 \text{ Kg m}^2$$
$$J_0 = 7 \cdot 10^{-6} \text{ Kg m}^2, \quad g = 9.8 \text{ m/s}^2, \quad k = 0.54 \text{ NSM}$$

We will apply Theorem 5, where $|\cdot|$ is a cubic vector norm.

Numerical calculations show that, for $d = 3.5$ and the matrix $\mathbf{K} = ||k_{ij}||$, having elements $k_{11} = k_{22} = -1.5$ and $k_{12} = k_{21} = 0$ for $s \geq 0$, the following inequality holds

$$d + \|-d\mathbf{I} + \mathbf{A}(s) - d^{-1}\mathbf{B}(s) + d^{-1}\mathbf{C}(s)\mathbf{K}\| + \gamma(-\mathbf{A}(s)) < -0.017$$

and consequently, with control (4.3), the programmed motion $\mathbf{q}_0(t)$ of the pendulum is exponentially stabilized.

Example 3. The problem of stabilizing the programmed motion of a two-link manipulator on a mobile base. We will assume that a two-link manipulator (Fig. 2) consisting of homogeneous hinged links with a load positioned in the grip of the second link moves in the

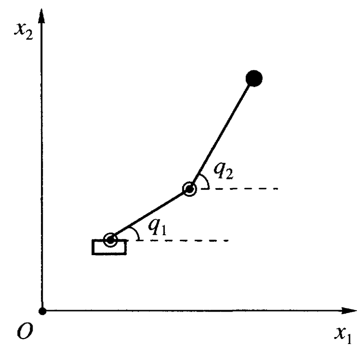

Fig. 2.

horizontal plane on a base performing translational motion.[12] We will write the kinetic energy of such a mechanical system

$$T = \frac{1}{2}\xi_1 l_1^2 \dot{q}_1^2 + \xi_2 l_2^2 \dot{q}_2^2 + \frac{1}{2}\xi_3 l_1 l_2 \cos(q_1 - q_2)\dot{q}_1\dot{q}_2 +$$
$$+ \frac{1}{2}(M_0 + m_1 + m_2 + m_3)(\dot{x}_1^2(t) + \dot{x}_2^2(t)) +$$
$$+ \xi_4 l_1(\dot{x}_2(t)\cos q_1 - \dot{x}_1(t)\sin q_1)\dot{q}_1 + \xi_3 l_2(\dot{x}_2(t)\cos q_2 - \dot{x}_1(t)\sin q_2)\dot{q}_2$$

Here

$$\xi_1 = \frac{m_1}{3} + m_2 + m_3, \quad \xi_2 = \frac{m_2}{3} + m_3, \quad \xi_3 = \frac{m_2}{2} + m_3, \quad \xi_4 = \frac{m_1}{2} + m_2 + m_3$$

m_1 and m_2 are the masses of the links, m_3 is the mass of the load, l_1 and l_2 are the lengths of the links, q_1 and q_2 are the hinged angles of the links, M_0 is the mass of the base and $x_1(t)$, $x_2(t)$ are the coordinates of the mobile base. Then the equation of controlled motion of the manipulator will take the form

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{P}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{W}(t, \mathbf{q})\mathbf{q} + \mathbf{R}(t, \mathbf{q}) = \mathbf{u} \tag{4.5}$$

where

$$\mathbf{H}(\mathbf{q}) = \begin{Vmatrix} \xi_1 l_1^2 & \xi_3 l_1 l_2 \cos(q_1 - q_2) \\ \xi_3 l_1 l_2 \cos(q_1 - q_2) & \xi_2 l_2^2 \end{Vmatrix}$$

$$\mathbf{P}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{Vmatrix} k & \xi_3 l_1 l_2 \sin(q_1 - q_2)\dot{q}_2 \\ -\xi_3 l_1 l_2 \sin(q_1 - q_2)\dot{q}_1 & k \end{Vmatrix}$$

$$\mathbf{W}(t, \mathbf{q}) = \begin{Vmatrix} -\xi_4 l_1 \ddot{x}_1(t) q_1^{-1} \sin q_1 & 0 \\ 0 & -\xi_4 l_2 \ddot{x}_1(t) q_2^{-1} \sin q_2 \end{Vmatrix}$$

$$\mathbf{R}(t, \mathbf{q}) = \begin{Vmatrix} \xi_4 l_1 \ddot{x}_2(t) \cos q_1 \\ \xi_3 l_1 \ddot{x}_2(t) \cos q_2 \end{Vmatrix}$$

$\mathbf{q} = (q_1, q_2)'$ is the vector of the hinged angles of the links, $\mathbf{u} = (u_1, u_2)'$ is the vector of controlling moments and k is the coefficient of the moments of the resistance forces, which are linear in the generalized velocities.

As in the previous example, we will apply Theorem 5, where $|\cdot|$ is a cubic vector norm. Numerical calculations were carried out with the following parameter values

$$m_1 = 2.5 \text{ Kg}, \quad m_2 = 1.5 \text{ Kg}, \quad m_3 = 0.4 \text{ Kg}, \quad l_1 = 0.5 \text{ m}, \quad l_2 = 0.8 \text{ m}$$

$$k = 15 \text{ NSM}, \quad \ddot{x}_1(t) = 0.1\cos(0.2t), \quad \ddot{x}_2(t) = 0.1\sin(0.3t)$$

Let the programmed trajectory have the form

$$\mathbf{q}_0(t) = \left\| \begin{matrix} \sin t \\ \pi/2 + 0.2\sin(0.2t) \end{matrix} \right\|$$

Then, if $d = 1.21$ and the matrix $\mathbf{K} = \|k_{ij}\|$ has the elements

$$k_{11} = 10.2, \quad k_{12} = 0.23, \quad k_{21} = -0.6, \quad k_{22} = 4.28$$

control (4.3) ensures exponential stabilization of the programmed trajectory $\mathbf{q}_0(t)$.

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